

BRACKET PRODUCTS FOR WEYL-HEISENBERG FRAMES

PETER G. CASAZZA AND M. C. LAMMERS

ABSTRACT. We provide a detailed development of a function valued inner product known as the bracket product and used effectively by de Boor, DeVore, Ron and Shen to study translation invariant systems. We develop a version of the bracket product specifically geared to Weyl-Heisenberg frames. This bracket product has all the properties of a standard inner product including Bessel's inequality, a Riesz Representation Theorem, and a Gram-Schmidt process which turns a sequence of functions (g_n) into a sequence (e_n) with the property that $(E_{mb}e_n)_{m,n \in \mathbb{Z}}$ is orthonormal in $L^2(\mathbb{R})$. Armed with this inner product, we obtain several results concerning Weyl-Heisenberg frames. First we see that fiberization in this setting takes on a particularly simple form and we use it to obtain a compressed representation of the frame operator. Next, we write down explicitly all those functions $g \in L^2(\mathbb{R})$ and $ab = 1$ so that the family $(E_{mb}T_{na}g)$ is complete in $L^2(\mathbb{R})$. One consequence of this is that for functions g supported on a half-line $[\alpha, \infty)$ (in particular, for compactly supported g), $(g, 1, 1)$ is complete if and only if $\sup_{0 \leq t < a} |g(t - n)| \neq 0$ a.e. Finally, we give a direct proof of a result hidden in the literature by proving: For any $g \in L^2(\mathbb{R})$, $A \leq \sum_n |g(t - na)|^2 \leq B$ is equivalent to $(E_{m/a}g)$ being a Riesz basic sequence.

1. INTRODUCTION

While working on some deep questions in non-harmonic Fourier series, Duffin and Schaeffer [14] introduced the notion of a frame for Hilbert spaces. Outside of this area, this idea seems to have been lost until Daubechies, Grossman and Meyer [12] brought attention to it in 1986. Duffin and Schaeffer's definition was an abstraction of a concept introduced by Gabor [17] in 1946 for doing signal analysis. Today the frames introduced by Gabor are called **Gabor frames** or **Weyl-Heisenberg frames**. Along with wavelets, Weyl-Heisenberg frames are still the backbone of modern day signal processing as well as a host of related topics.

In the study of shift invariant systems and frames several authors, including de Boor, DeVore, Ron and Shen [2, 3, 25, 26], have made extensive use of the

1991 *Mathematics Subject Classification*. Primary: 42A65, 42C15, 42C30.

Key words and phrases. Weyl-Heisenberg (Gabor) frames, bracket products.

The first author was supported by NSF DMS 970618.

so called **bracket product**

$$[f, g](x) = \sum_{\beta \in 2\pi^d} f(x + \beta) \overline{g(x + \beta)}.$$

One may view this bracket product as a pointwise inner product and we will refer to it as such throughout the paper. In what follows we give a more thorough development of the bracket product itself and its application to univariate principal Weyl-Heisenberg systems. We hope that our development of the bracket product will aid in applying it to Weyl-Heisenberg systems as well as other areas where shift-invariance is of importance. Because we would like to be able to change the shift parameter from 2π to arbitrary $a \in \mathbb{R}^+$ we will refer to this bracket product as the **a-inner product**.

Let us briefly discuss the organization of the paper. In Section 2 we review the notation and terminology, as well as the basic results of Weyl-Heisenberg frames. In Section 3 we ever so slightly alter the definition of bracket product to get the a-inner product and develop its basic properties. In section 4 we discuss orthogonality with respect to the a -inner product and develop such notions as orthonormal sequences, orthonormal bases and a Bessel's inequality all with respect to the a-inner product.. In Section 5 we study a-factorable operators. These are the natural bounded linear operators related to the a-inner product. We will prove that the a-inner product has a Riesz Representation Theorem for a-factorable operators. In Section 6 we will relate our a-inner product directly to Weyl-Heisenberg frames. We will see that this gives a representation for the frame operator for a Weyl-Heisenberg frame (g, a, b) in terms of the $1/b$ -inner product. This representation can be viewed as a simple form of fiberization technique developed by Ron and Shen [25, 26]. In Section 7 we use these ideas to prove two theorems concerning Weyl-Heisenberg frames. The first is "half" of a result proved independently by Daubechies, H. Landau, Z. Landau [13] ; Janssen [21]; and by Ron and Shen [26]. The second is a complete listing of all functions $g \in L^2(\mathbb{R})$ and $ab = 1$ so that the Weyl-Heisenberg system is complete. A surprising consequence of this is that for a function supported on a half line, the minimal necessary condition for completeness $\sup_n |g(t - na)| \neq 0$ a.e. becomes sufficient. In Section 8 we see that the a-inner product gives a natural definition for an a-frame, and that these frames are a natural generalization of regular frames. In particular, we show that (g, a, b) is a WH-frame iff the translates of g , (g, a) , forms a $(1/b)$ -frame. We will also look at a-Riesz bases and their relationship to Riesz bases for a Hilbert space. Finally, in Section 9 we show that the Gram-Schmidt orthogonalization procedure works exactly as expected to produce a-orthonormal sequences with the proper spans.

The authors would like to thank A.J.E.M. Janssen for his helpful comments. In particular we would like to thank him for pointing out the connection between what we refer to as the compression of the frame operator and fiberization. Also, we would like to thank R. DeVore and A. Ron for useful discussions concerning the material in this paper.

2. PRELIMINARIES

We use $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ to denote the natural numbers, integers, real numbers and complex numbers, respectively. A **scalar** is an element of \mathbb{R} or \mathbb{C} . Integration is always with respect to Lebesgue measure. $L^2(\mathbb{R})$ will denote the complex Hilbert space of square integrable functions mapping \mathbb{R} into \mathbb{C} . A bounded unconditional basis for a Hilbert space H is called a **Riesz basis**. That is, (f_n) is a Riesz basis for H if and only if there is an orthonormal basis (e_n) for H and an operator $T : H \rightarrow H$ defined by $T(e_n) = f_n$, for all n . We call (f_n) a **Riesz basic sequence** if it is a Riesz basis for its closed linear span. For $E \subset H$, we write $\text{span } E$ for the **closed linear span of E**.

In 1952, Duffin and Schaeffer [14] were working on some deep problems in non-harmonic Fourier series. This led them to define

Definition 2.1. *A sequence $(f_n)_{n \in \mathbb{Z}}$ of elements of a Hilbert space H is called a **frame** if there are constants $A, B > 0$ such that*

$$(2.1) \quad A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The numbers A, B are called the **lower** and **upper frame bounds** respectively. The largest number $A > 0$ and smallest number $B > 0$ satisfying the frame inequalities for all $f \in H$ are called the **optimal frame bounds**. The frame is a **tight frame** if $A = B$ and a **normalized tight frame** if $A = B = 1$. A frame is **exact** if it ceases to be a frame when any one of its elements is removed. It is known that a frame is exact if and only if it is a Riesz basis. A non-exact frame is called **over-complete** in the sense that at least one vector can be removed from the frame and the remaining set of vectors will still form a frame for H (but perhaps with different frame bounds). If $f_n \in H$, for all $n \in \mathbb{Z}$, we call $(f_n)_{n \in \mathbb{Z}}$ a **frame sequence** if it is a frame for its closed linear span in H .

We will consider frames from the operator theoretic point of view. To formulate this approach, let (e_n) be an orthonormal basis for an infinite dimensional Hilbert space H and let $f_n \in H$, for all $n \in \mathbb{Z}$. We call the operator $T : H \rightarrow H$ given by $Te_n = f_n$ the **preframe operator** associated with (f_n) . Now, for

each $f \in H$ and $n \in \mathbb{Z}$ we have $\langle T^*f, e_n \rangle = \langle f, Te_n \rangle = \langle f, f_n \rangle$. Thus

$$(2.2) \quad T^*f = \sum_n \langle f, f_n \rangle e_n, \quad \text{for all } f \in H.$$

By (2.2)

$$\|T^*f\|^2 = \sum_n |\langle f, f_n \rangle|^2, \quad \text{for all } f \in H.$$

It follows that the preframe operator is bounded if and only if (f_n) has a finite upper frame bound B . Comparing this to Definition 2.1 we have

Theorem 2.2. *Let H be a Hilbert space with an orthonormal basis (e_n) . Also let (f_n) be a sequence of elements of H and let $Te_n = f_n$ be the preframe operator. The following are equivalent:*

- (1) (f_n) is a frame for H .
- (2) The operator T is bounded, linear and onto.
- (3) The operator T^* is an (possibly into) isomorphism called the **frame transform**.

Moreover, (f_n) is a normalized tight frame if and only if the preframe operator is a quotient map (i.e. a co-isometry).

The dimension of the kernel of T is called the **excess** of the frame. It follows that $S = TT^*$ is an invertible operator on H , called the **frame operator**. Moreover, we have

$$Sf = TT^*f = T\left(\sum_n \langle f, f_n \rangle e_n\right) = \sum_n \langle f, f_n \rangle Te_n = \sum_n \langle f, f_n \rangle f_n.$$

A direct calculation now yields

$$\langle Sf, f \rangle = \sum_n |\langle f, f_n \rangle|^2.$$

Therefore, the **frame operator is a positive, self-adjoint invertible operator** on H . Also, the frame inequalities (2.1) yield that (f_n) is a frame with frame bounds $A, B > 0$ if and only if $A \cdot I \leq S \leq B \cdot I$. Hence, (f_n) is a normalized tight frame if and only if $S = I$. Also, a direct calculation yields

$$(2.3) \quad \begin{aligned} f = SS^{-1}f &= \sum_n \langle S^{-1}f, f_n \rangle f_n \\ &= \sum_n \langle f, S^{-1}f_n \rangle f_n \\ &= \sum_n \langle f, S^{-1/2}f_n \rangle S^{-1/2}f_n. \end{aligned}$$

We call $(\langle S^{-1}f, f_n \rangle)$ the **frame coefficients** for f . One interpretation of equation (2.3) is that $(S^{-1/2}f_n)$ is a normalized tight frame.

Theorem 2.3. *Every frame (f_n) (with frame operator S) is equivalent to the normalized tight frame $(S^{-1/2}f_n)$.*

We will work here with a particular class of frames called Weyl-Heisenberg frames. To formulate these frames, we first need some notation. For a function f on \mathbb{R} we define the operators:

$$\begin{aligned} \text{Translation: } T_a f(x) &= f(x - a), & a \in \mathbb{R} \\ \text{Modulation: } E_a f(x) &= e^{2\pi i a x} f(x), & a \in \mathbb{R} \\ \text{Dilation: } D_a f(x) &= |a|^{-1/2} f(x/a), & a \in \mathbb{R} - \{0\} \end{aligned}$$

We also use the symbol E_a to denote the **exponential function** $E_a(x) = e^{2\pi i a x}$. Each of the operators T_a, E_a, D_a are unitary operators on $L^2(\mathbb{R})$ and they satisfy:

$$\begin{aligned} T_a E_b f(x) &= e^{2\pi i b(x-a)} f(x - a); \\ E_b T_a f(x) &= e^{2\pi i b x} f(x - a); \\ D_a T_b f(x) &= |a|^{-1/2} f\left(\frac{x}{a} - b\right); \\ T_b D_a f(x) &= |a|^{-1/2} f\left(\frac{x-b}{a}\right); \\ E_b D_a f(x) &= e^{2\pi i b x} |a|^{-1/2} f\left(\frac{x}{a}\right); \\ D_a E_b f(x) &= e^{2\pi i b x/a} |a|^{-1/2} f\left(\frac{x}{a}\right). \end{aligned}$$

In 1946 Gabor [17] formulated a fundamental approach to signal decomposition in terms of elementary signals. This method resulted in **Gabor frames** or as they are often called today **Weyl-Heisenberg frames**.

Definition 2.4. *If $a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ we call $(E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$ a **Weyl-Heisenberg system** (**WH-system** for short) and denote it by (g, a, b) . We denote by (g, a) the family $(T_{na}g)_{n \in \mathbb{Z}}$. We call g the **window function**.*

If the WH-system (g, a, b) forms a frame for $L^2(\mathbb{R})$, we call this a **Weyl-Heisenberg frame** (**WH-frame** for short). The numbers a, b are the **frame parameters** with a being the **shift parameter** and b being the **modulation parameter**. We will be interested in when there are finite upper frame bounds for a WH-system. We call this class of functions the **preframe functions** and denote this class by **PF**. It is easily checked that

Proposition 2.5. *The following are equivalent:*

- (1) $g \in \mathbf{PF}$.
- (2) The operator

$$Sf = \sum_n \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}g,$$

is a well defined bounded linear operator on $L^2(\mathbb{R})$.

We will need the WH-frame identity due to Daubechies [10]. To simplify the notation a little we introduce the following auxiliary functions defined for a $g \in L^2(\mathbb{R})$ and all $k \in \mathbb{Z}$ by

$$G_k(t) = \sum_{n \in \mathbb{Z}} g(t - na) \overline{g(t - na - k/b)}.$$

In particular,

$$G_0(t) = \sum_{n \in \mathbb{Z}} |g(t - na)|^2.$$

Theorem 2.6. (WH-Frame Identity.) *If $\sum_n |g(t - na)|^2 \leq B$ a.e. and $f \in L^2(\mathbb{R})$ is bounded and compactly supported, then*

$$\sum_{n, m \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = F_1(f) + F_2(f),$$

where

$$F_1(f) = b^{-1} \int_{\mathbb{R}} |f(t)|^2 G_0(t) dt,$$

and

$$\begin{aligned} F_2(f) &= b^{-1} \sum_{k \neq 0} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt \\ &= b^{-1} \sum_{k \geq 1} 2 \operatorname{Re} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt. \end{aligned}$$

There are many restrictions on the g, a, b in order that (g, a, b) form a WH-frame. We will make use of a few of them here. The first is a simple application of the WH-frame Identity. That is, if we put functions supported on $[0, 1/b]$ into this identity, then $F_2(f) = 0$. Now the WH-frame Identity combined with the frame condition quickly yields,

Theorem 2.7. *If (g, a, b) is a WH-frame with frame bounds A, B then*

$$A \leq b G_0(t) \leq B, \quad \text{a.e.}$$

Casazza and Christensen [6] noted that we have a similar upper bound condition with a replaced by $1/b$.

Proposition 2.8. *If (g, a, b) is a WH-frame with upper frame bound B then*

$$\sum_{n \in \mathbb{Z}} |g(t - n/b)|^2 \leq B, \quad \text{a.e..}$$

There are also some restrictions on a, b for (g, a, b) to be a frame.

Proposition 2.9. *Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$.*

- (1) *If $(E_{mb}T_{na}g)$ is complete, then $ab \leq 1$.*
- (2) *If (g, a, b) is a WH-frame and*
 - (i) *$ab < 1$ then (g, a, b) is over-complete.*
 - (ii) *$ab = 1$ then (g, a, b) is a Riesz basis.*

Part (1) of Proposition 2.9 has a complicated history (see [10] for a discussion) which derives from the work of Rieffel [24]. Today, there is a simpler proof using Beurling density due to Ramanathan and Steger [23]. Moreover, the results of Ramanathan and Steger [23] combined with an important example of Benedetto, Heil and Walnut [1] shows that the form of the lattice in the Rieffel result [24] is quite important to the conclusion. There are many derivations available for (2) [7, 10, 11, 18, 20, 21].

A recent very important result was proved independently by Daubechies, H. Landau and Z. Landau [13], Janssen [21], and Ron and Shen [26].

Theorem 2.10. *For $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, the following are equivalent:*

- (1) *(g, a, b) is a WH-frame.*
- (2) *The family $(E_{m/a}T_{n/b}g)_{m,n \in \mathbb{Z}}$ is a Riesz basic sequence in $L^2(\mathbb{R})$.*

Ron and Shen attained this result with a technique they call **Gramian analysis**. At the heart of this technique is the Gramian matrix \mathcal{G} which is used to decompose the pre-frame operator and its adjoint. The tie in with the bracket product becomes clear when one sees that in the shift-invariant case (i.e. consider only $(T_{na}g)$ this matrix becomes $\mathcal{G} = [g, g]$.

Finally, we will need the classification of tight WH-frames. Parts of this are due to various authors. A direct proof from the definitions as well as the historical development can be found in [7].

Theorem 2.11. *Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. The following are equivalent:*

- (1) *$(E_{mb}T_{na}g)$ is a normalized tight Weyl-Heisenberg frame for $L^2(\mathbb{R})$.*
- (2) *We have:*
 - (a) $G_0(t) = \sum_{n \in \mathbb{Z}} |g(t - na)|^2 = b$ a.e.
 - (b) *For all $k \neq 0$, $G_k(t) = \sum_n g(t - na) \overline{g(t - na - k/b)} = 0$ a.e.*
- (3) *We have $g \perp E_{n/a}T_{m/b}g$, for all $(n, m) \neq (0, 0)$ and $\|g\|^2 = ab$.*
- (4) *$(E_{n/a}T_{m/b}g)$ is an orthogonal sequence in $L^2(\mathbb{R})$ and $\|g\|^2 = ab$.*
- (5) *$(E_{mb}T_{na}g)$ is a Weyl-Heisenberg frame for $L^2(\mathbb{R})$ with frame operator S and $Sg = g$.*

Moreover, when at least one of (1) – (5) holds, $(E_{mb}T_{na}g)$ is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $\|g\| = 1$.

We next recall the **Wiener amalgam space** $W(L^\infty, L^1)$ which consists of all functions g so that for some $a > 0$ we have,

$$\|g\|_{W,a} = \sum_{n \in \mathbb{Z}} \|g \cdot \chi_{[an, a(n+1))}\|_\infty = \sum_{n \in \mathbb{Z}} \|T_{na} \cdot \chi_{[0,a)}\|_\infty < \infty.$$

It is easily checked that $W(L^\infty, L^1)$ is a Banach space with the above norm. Also, if $\|g\|_{W,a} < \infty$, for one $a > 0$, then this norm is finite for all $a > 0$.

3. POINTWISE INNER PRODUCTS

A number of the basic results in this section can be found in various other papers [2, 3, 25, 26]. For the sake of completeness, and to create a good reference for this inner product we present them here. To guarantee that our inner product is well defined, we need to first check some convergence properties for elements of $L^2(\mathbb{R})$.

Proposition 3.1. *For $f, g \in L^2(\mathbb{R})$ and $a \in \mathbb{R}$ the series*

$$\sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na)}$$

converges unconditionally a.e. to a function in $L^1[0, a]$.

Proof. If $f, g \in L^2(\mathbb{R})$ then $fg \in L^1(\mathbb{R})$. Hence,

$$\begin{aligned} \|fg\|_{L^1} &= \int_{\mathbb{R}} |f(t) \overline{g(t)}| dt \\ &= \sum_{n \in \mathbb{Z}} \int_0^a |f(t - na) \overline{g(t - na)}| dt \\ &= \int_0^a \sum_{n \in \mathbb{Z}} |f(t - na) \overline{g(t - na)}| dt < \infty. \end{aligned}$$

The last inequality follows by the Monotone Convergence Theorem. This yields both the interchange of the integral and the sum and the existence of $\sum f(t - na) \overline{g(t - na)}$ as a function in $L^1[0, a]$. \square

A simple application of the Lebesgue Dominated Convergence Theorem combined with Proposition 3.1 yields,

Corollary 3.2. *For all $f, g \in L^2(\mathbb{R})$ we have*

$$\langle f, g \rangle = \int_0^a \sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na)} dt.$$

Now we introduce the pointwise inner product for WH-frames. We can also view this as a vector-valued inner product.

Definition 3.3. Fix $a \in \mathbb{R}$. For all $f, g \in L^2(\mathbb{R})$ we define the **a-pointwise inner product of f and g** (called the **a-inner product** for short) by

$$\langle f, g \rangle_a(t) = \sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na)}, \quad \text{for all } t \in \mathbb{R}.$$

We define the **a-norm of f** by

$$\|f\|_a(t) = \sqrt{\langle f, f \rangle_a(t)}.$$

We emphasize here that the a-inner product and the a-norm are *functions* on \mathbb{R} which are clearly a-periodic. To cut down on notation, whenever we have an a-periodic function on \mathbb{R} , we will also consider it a function on $[0, a]$. The convergence of these series is guaranteed by our earlier discussion. In fact, the a-inner product $\langle \cdot, \cdot \rangle_a$ is a mapping from $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ to the a-periodic functions on \mathbb{R} whose restriction to $[0, a]$ lie in $L^1[0, a]$.

First we show that the a-inner product really is a good generalization of the standard notion of inner products for a Hilbert space.

Theorem 3.4. Let $f, g, h \in L^2(\mathbb{R})$, $c, d \in \mathbb{C}$, and $a, b \in \mathbb{R}$. The following properties hold:

(1) $\langle f, g \rangle_a$ is a periodic function of period a on \mathbb{R} with $\langle f, g \rangle_a \in L^1[0, a]$.

(2) We have

$$\|f\|_{L^2(\mathbb{R})} = \left\| \|f\|_a(t) \right\|_{L^2[0, a]}.$$

(3) We have

$$\langle f, g \rangle = \int_0^a \langle f, g \rangle_a(t) dt.$$

(4) $\langle cf + dg, h \rangle_a = c \langle f, h \rangle_a + d \langle g, h \rangle_a$.

(5) $\langle f, cg + dh \rangle_a = \bar{c} \langle f, g \rangle_a + \bar{d} \langle f, h \rangle_a$.

(6) $\langle f, g \rangle_a = \overline{\langle g, f \rangle_a}$.

(7) $\langle fg, h \rangle_a = \langle f, \bar{g}h \rangle_a$.

(8) If $\langle f, g \rangle_a = 0$ then $\langle f, g \rangle = 0$.

(9) $\langle T_b f, T_b g \rangle_a = T_b \langle f, g \rangle_a$.

$$(10) \|T_b g\|_a^2 = T_b \|g\|_a^2$$

$$(11) \langle T_b f, g \rangle_a = T_b \langle f, T_{-b} g \rangle_a.$$

$$(12) \langle f, g \rangle_a = \frac{1}{\sqrt{ab}} D_{ab} \langle D_{\frac{1}{ab}} f, D_{\frac{1}{ab}} g \rangle_{\frac{1}{b}}.$$

Proof. All the proofs follow directly from the definitions. We will give a small sample to show how they proceed.

(3) This is just Corollary 3.2.

(4) We calculate:

$$\begin{aligned} \langle cf + dg, h \rangle_a(t) &= \sum_{n \in \mathbb{Z}} [cf + dg](t - na) \overline{h(t - na)} \\ &= c \sum_{n \in \mathbb{Z}} f(t - na) \overline{h(t - na)} + d \sum_{n \in \mathbb{Z}} g(t - na) \overline{h(t - na)} \\ &= c \langle f, h \rangle_a + d \langle g, h \rangle_a. \end{aligned}$$

(8) If $\langle f, g \rangle_a = 0$ then by (3),

$$\langle f, g \rangle = \int_0^a \langle f, g \rangle_a(t) dt = 0.$$

(11) Again, we calculate

$$\begin{aligned} \langle T_b f, g \rangle_a &= \sum_{n \in \mathbb{Z}} f(t - b - na) \overline{g(t - na)} \\ &= T_b \sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na + b)} = T_b \langle f, T_{-b} g \rangle_a. \end{aligned}$$

(12) We compute,

$$\begin{aligned} \langle D_{\frac{1}{ab}} f, D_{\frac{1}{ab}} g \rangle_{\frac{1}{b}}(t) &= \langle \sqrt{ab} f(ab \cdot), \sqrt{ab} g(ab \cdot) \rangle_{\frac{1}{b}}(t) \\ &= ab \sum_{n \in \mathbb{Z}} f(ab(t - n/b)) \overline{g(ab(t - n/b))} \\ &= ab \sum_{n \in \mathbb{Z}} f(abt - na) \overline{g(abt - na)} \\ &= ab \langle f, g \rangle_a(abt) = \sqrt{ab} D_{\frac{1}{ab}} \langle f, g \rangle_a(t). \end{aligned}$$

□

Once one sees what is going on, it is not difficult to mimic the standard proofs for the usual inner product on a Hilbert space to obtain the following results for the a -inner product.

Proposition 3.5. *For all $f, g \in L^2(\mathbb{R})$ we have,*

$$(1) \quad | \langle f, g \rangle_a | \leq \|f\|_a \|g\|_a, \text{ a.e.}$$

$$(2) \quad \|f + g\|_a^2 = \|f\|_a^2 + 2\operatorname{Re} \langle f, g \rangle_a + \|g\|_a^2.$$

$$(3) \quad \|f + g\|_a \leq \|f\|_a + \|g\|_a.$$

$$(4) \quad \|f + g\|_a^2 + \|f - g\|_a^2 = 2(\|f\|_a^2 + \|g\|_a^2), \text{ a.e.}$$

Since our a -inner product is an a -periodic function, it enjoys some special properties related to a -periodic functions.

Proposition 3.6. *Let $f, g \in L^2(\mathbb{R})$ and let $h \in L^\infty(\mathbb{R})$ be an a -periodic function. Then*

$$\langle fh, g \rangle_a = h \langle f, g \rangle_a \quad \text{and} \quad \langle f, hg \rangle_a = \overline{h} \langle f, g \rangle_a.$$

In particular, if h satisfies $h(t) \neq 0$ a.e., then $\langle f, g \rangle_a = 0$ if and only if $\langle fh, g \rangle_a = \langle f, g\overline{h} \rangle_a = 0$.

Proof. We compute

$$\begin{aligned} \langle fh, g \rangle_a(t) &= \sum_{n \in \mathbb{Z}} f(t - na) h(t - na) \overline{g(t - na)} \\ &= \sum_{n \in \mathbb{Z}} f(t - na) h(t) \overline{g(t - na)} \\ &= h(t) \sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na)} = h(t) \langle f, g \rangle_a(t). \end{aligned}$$

□

Next we normalize our functions in the a -inner product. For $f \in L^2(\mathbb{R})$, we define the **a -pointwise normalization of f** to be

$$N_a(f)(t) = \begin{cases} \frac{f(t)}{\|f\|_a(t)} : & \|f\|_a(t) \neq 0 \\ 0 : & \|f\|_a(t) = 0. \end{cases}$$

We now have

Proposition 3.7. *Let $f, g \in L^2(\mathbb{R})$.*

(1) We have

$$\langle N_a(f), g \rangle_a = \frac{\langle f, g \rangle_a}{\|f\|_a}, \quad \text{where } \|f\|_a \neq 0.$$

In particular, $\langle f, g \rangle_a = 0$ if and only if $\langle N_a(f), g \rangle_a = 0$.

(2) For $f \neq 0$ a.e. we have

$$\langle N_a(f), N_a(f) \rangle_a(t) = \sum_{n \in \mathbb{Z}} |N_a(f)(t - na)|^2 = 1, \text{ a.e.}$$

(3) we have

$$\|N_a(f)\|_{L^2(\mathbb{R})}^2 = \lambda(\text{supp } \|f\|_a|_{[0,a]}) \leq a.$$

where λ denotes Lebesgue measure.

(4) $N_a(N_a(f)) = N_a(f)$.

Proof. (1) We compute

$$\langle N_a(f), g \rangle_a = \sum_{n \in \mathbb{Z}} N_a(f)(t - na) \overline{g(t - na)} = \sum_{n \in \mathbb{Z}} \frac{f(t - na)}{\|f\|_a(t - na)} \overline{g(t - na)}.$$

Since our inner product is a -periodic, this equality becomes,

$$\frac{1}{\|f\|_a(t)} \sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na)} = \frac{\langle f, g \rangle_a(t)}{\|f\|_a(t)}, \quad \text{where } \|f\|_a(t) \neq 0.$$

(2) This is two applications of part (1).

(3) By (2) we have

$$\begin{aligned} \|N_a(f)\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |N_a(f)(t)|^2 dt \\ &= \int_0^a \sum_{n \in \mathbb{Z}} |N_a(f)(t - na)|^2 dt = \int_0^a \mathbf{1}_{\text{supp } \|f\|_a}(t) dt \leq a. \end{aligned}$$

(4) This is immediate from (2). □

4. a -ORTHOGONALITY

The notion of orthogonality with respect to the a -inner product has been used primarily to describe the orthogonal complement in the usual inner product for shift-invariant spaces. In this section we explore more thoroughly what it means to be a -orthogonal and develop such things as a -orthonormal sequences and a Bessel inequality for the a -inner product. This property gives one of the main applications of the a -inner product in Weyl-Heisenberg frame theory. For as we will see, orthogonality in this form is very strong.

Definition 4.1. For $f, g \in L^2(\mathbb{R})$, we say that f and g are **a -orthogonal**, and write $f \perp_a g$, if $\langle f, g \rangle_a = 0$. We define the **a -orthogonal complement** of $E \subset L^2(\mathbb{R})$ by

$$E^{\perp_a} = \{g : \langle f, g \rangle_a = 0, \text{ for all } f \in E\}.$$

Similarly, an **a-orthogonal sequence** is a sequence (f_n) satisfying $f_n \perp_a f_m$, for all $n \neq m$. This is an **a-orthonormal sequence** if we also have $\|f\|_a = 1$, a.e. where $\|f\|_a \neq 0$.

We now identify an important class of functions for working with the a-inner product.

Definition 4.2. We say that $g \in L^2(\mathbb{R})$ is **a-bounded** if there is a $B > 0$ so that

$$| \langle g, g \rangle_a | \leq B, \quad \text{a.e.}$$

We let $L_a^\infty(\mathbb{R})$ denote the family of a-bounded functions.

We have that $L_a^\infty(\mathbb{R})$ is a non-closed linear subspace of $L^\infty(\mathbb{R})$. To see this, first observe that $L_a^\infty(\mathbb{R})$ is just the family of functions $g \in L^2(\mathbb{R})$ for which $\|g\|_a$ is bounded. So by the properties we have developed for $\|\cdot\|_a$ we have that $L_a^\infty(\mathbb{R})$ is a subspace of $L^\infty(\mathbb{R})$. Since $L_a^\infty(\mathbb{R})$ contains all bounded compactly supported functions in $L^2(\mathbb{R})$, and it is easily seen to not equal $L^2(\mathbb{R})$, we have that this is a non-closed subspace. Note also that the Wiener amalgam space is a subspace of $L_a^\infty(\mathbb{R})$.

We have not defined orthonormal bases for the a-inner product yet since, as we will see, this requires a little more care. First we need to develop the basic properties of a-orthogonality.

Proposition 4.3. If $E \subset L^2(\mathbb{R})$,

$$E^{\perp_a} = \cap_{\phi \in L_a^\infty(\mathbb{R})} (\phi E)^\perp = (\text{span}_{\phi \in L_a^\infty(\mathbb{R})} \phi E)^\perp.$$

Proof. Let $f \in E^{\perp_a}$. For any $g \in E$ and any a-periodic function $\phi \in L_a^\infty(\mathbb{R})$ we have by Proposition 3.6

$$\langle f, \phi g \rangle_a = \overline{\phi} \langle f, g \rangle_a = 0.$$

Hence, $f \perp_a \phi g$. That is, $f \in (\phi E)^\perp$.

Now let $f \in \cap (\phi E)^\perp$, the intersection being taken over all bounded a-periodic ϕ . Let $g \in E$ and define for $n \in \mathbb{N}$,

$$\phi_n(t) = \begin{cases} \langle f, g \rangle_a(t) : & |\langle f, g \rangle_a(t)| \leq n \\ 0 : & \text{otherwise.} \end{cases}$$

Note that ϕ_n is a-periodic. Now we compute,

$$\begin{aligned} 0 = \langle f, \phi_n g \rangle &= \int_{\mathbb{R}} f(t) \overline{\phi_n(t)} g(t) dt \\ &= \int_0^a \left(\sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na)} \right) \overline{\phi_n(t)} dt \\ &= \int_0^a \langle f, g \rangle_a(t) \overline{\phi_n(t)} dt = \int_0^a |\phi_n(t)|^2 dt. \end{aligned}$$

Therefore, $\phi_n = 0$, for all $n \in \mathbb{Z}$. Hence, $\langle f, g \rangle_a = 0$, and so $f \perp_a g$. That is, $f \perp_a E$. \square

By Theorem 3.4 (8), we have that $E^{\perp_a} \subset E^{\perp}$.

Corollary 4.4. *For $E \subset L^2(\mathbb{R})$, E^{\perp_a} is a norm closed linear subspace of E^{\perp} .*

The next result which can be found in [2] shows more clearly what orthogonality means in this setting .

Proposition 4.5. *For $f, g \in L^2(\mathbb{R})$, the following are equivalent:*

- (1) $f \perp_a g$.
- (2) $\text{span}_{m \in \mathbb{Z}} E_{m/a} f \perp \text{span}_{m \in \mathbb{Z}} E_{m/a} g$.

Proof. Fix $m \in \mathbb{Z}$ and compute

$$\langle f, E_{m/a} g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t)} e^{-2\pi i(m/a)t} dt = \int_0^a \langle f, g \rangle_a(t) e^{-2\pi i(m/a)t} dt.$$

It follows that $\langle f, E_{m/a} g \rangle = 0$, for all $m \in \mathbb{Z}$ if and only if $\langle f, g \rangle_a = 0$. A moment's reflection should convince the reader that this is all we need. \square

Definition 4.6. *We say that $E \subset L^2(\mathbb{R})$ is an **a-periodic closed set** if for any $f \in E$ and any $\phi \in L_a^\infty(\mathbb{R})$ we have that $\phi f \in E$.*

The next result follows immediately from Propositions 4.5 and 4.3.

Corollary 4.7. *For any $E \subset L^2(\mathbb{R})$, E^{\perp_a} is an a-periodic closed set. If E is an a-periodic closed set then $E^{\perp} = E^{\perp_a}$.*

Now we observe what orthogonality means for $(E_{m/a} g)$ in terms of the regular inner product.

Proposition 4.8. *If $g \in L^2(\mathbb{R})$ and $\|g\|_a = 1$ a.e., then $(\frac{1}{\sqrt{a}} E_{m/a} g)_{m \in \mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.*

Proof. For any $n, m \in \mathbb{Z}$ we have

$$\begin{aligned} \langle E_{n/a} g, E_{m/a} g \rangle &= \int_{\mathbb{R}} |g(t)|^2 e^{2\pi i[(n-m)/a]t} dt \\ &= \int_0^a \|g\|_a^2(t) e^{2\pi i[(n-m)/a]t} dt \\ &= \int_0^a e^{2\pi i[(n-m)/a]t} dt = a \delta_{nm}. \end{aligned}$$

\square

Corollary 4.9. *If $(g_n)_{n \in \mathbb{N}}$ is an a -orthonormal sequence in $L^2(\mathbb{R})$, then $(E_{m/a}g_n)_{n,m \in \mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.*

Proof. We need that for all $(n, m) \neq (\ell, k) \in \mathbb{Z} \times \mathbb{Z}$, $E_{m/a}g_n \perp E_{\ell/a}g_k$. But, if $m \neq k$, this is Proposition 4.5, and if $m = k$, this is Proposition 4.8. \square

Corollary 4.9 tells us how to define an a -orthonormal basis.

Definition 4.10. *Let $g_n \in L^2(\mathbb{R})$. We call (g_n) an **a -orthonormal basis** for $L^2(\mathbb{R})$ if it is an a -orthonormal sequence and*

$$\overline{\text{span}} (E_{m/a}g_n)_{n,m \in \mathbb{Z}} = L^2(\mathbb{R}).$$

Proposition 4.11. *A sequence (g_n) in $L^2(\mathbb{R})$ is an a -orthonormal basis if and only if $(E_{m/a}g_n)_{n,m \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

We would like to capture the important Bessel's Inequality for a -orthonormal sequences but before we do so we need to insure that $\langle f, g \rangle_a g$ remains an $L^2(\mathbb{R})$ for functions $g \in L_a^\infty(\mathbb{R})$.

Proposition 4.12. *If $g \in L_a^\infty(\mathbb{R})$ then $\langle f, g \rangle_a g \in L^2(\mathbb{R})$ for all $f \in L^2(\mathbb{R})$.*

Proof First we need to show $\langle f, g \rangle_a \in L^2([0, a])$. This follows from the Cauchy-Schwarz inequality for the a -inner product.

$$\begin{aligned} \|\langle f, g \rangle_a\|_{L^2[0,a]}^2 &= \int_0^a |\langle f, g \rangle_a(t)|^2 dt \\ &\leq \int_0^a \langle f, f \rangle_a(t) \langle g, g \rangle_a(t) dt \\ &\leq B \int_0^a \langle f, f \rangle_a(t) dt = B \|f\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Now we can get our results which follows from the Monotone convergence theorem and the result above.

$$\begin{aligned} \|\langle f, g \rangle_a g\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |\langle f, g \rangle_a(t)g(t)|^2 dt \\ &= \sum \int_0^a |\langle f, g \rangle_a(t)|^2 |g(t - na)|^2 dt \\ &\leq \int_0^a |\langle f, g \rangle_a(t)|^2 \langle g, g \rangle_a(t) dt \\ &\leq B^2 \|f\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

\square

Theorem 4.13. *If $(g_n)_{n \in \mathbb{N}}$ is an a -orthonormal sequence in $L^2(\mathbb{R})$, then for all $f \in L^2(\mathbb{R})$ we have that*

(1) the series of functions $\sum_{n \in \mathbb{N}} \langle f, g_n \rangle_a g_n$ converges in $L^2(\mathbb{R})$.

(2) We have “Bessel’s Inequality”,

$$\langle f, f \rangle_a \geq \sum_{n=1}^{\infty} |\langle f, g_n \rangle_a|^2.$$

Note that this is an inequality for functions.

Moreover, if $f \in \text{span} (E_{m/a} g_n)_{n \in \mathbb{Z}}$, then

$$\langle f, f \rangle_a = \sum_{n=1}^{\infty} |\langle f, g_n \rangle_a|^2.$$

Proof. Fix $1 \leq m$ and let

$$h = \sum_{n=1}^m \langle f, g_n \rangle_a g_n.$$

Using the fact that the a -inner product of two functions is a -periodic (and hence may be factored out of the a -inner product) we have

$$\begin{aligned} \langle h, h \rangle_a &= \left\langle \sum_{n=1}^m \langle f, g_n \rangle_a g_n, \sum_{k=1}^m \langle f, g_k \rangle_a g_k \right\rangle_a \\ &= \sum_{n,k=1}^m \langle f, g_n \rangle_a \overline{\langle f, g_k \rangle_a} \langle g_n, g_k \rangle_a \\ &= \sum_{n=1}^m |\langle f, g_n \rangle_a|^2. \end{aligned}$$

Letting $g = f - h$ we have by the same type of calculation as above,

$$\begin{aligned} \langle h, g \rangle_a &= \left\langle \sum_{n=1}^m \langle f, g_n \rangle_a g_n, f - \sum_{k=1}^m \langle f, g_k \rangle_a g_k \right\rangle_a \\ &= \sum_{n=1}^m |\langle f, g_n \rangle_a|^2 - \sum_{k=1}^m |\langle f, g_k \rangle_a|^2 = 0. \end{aligned}$$

So we have decomposed f into two a -orthogonal functions h, g . Therefore,

$$\begin{aligned}
\langle f, f \rangle_a &= \langle h + g, h + g \rangle_a \\
&= \langle h, h \rangle_a + \langle g, g \rangle_a \\
&= \sum_{n=1}^m |\langle f, g_n \rangle_a|^2 + \langle g, g \rangle_a \geq \sum_{n=1}^m |\langle f, g_n \rangle_a|^2.
\end{aligned}$$

Since m was arbitrary, we have (2) of the Theorem. For (1), we just put together what we know. By (2) and the Monotone Convergence Theorem, we have that the series of functions $\sum_{n \in \mathbb{N}} |\langle f, g_n \rangle_a|^2$ converges in $L^1[0, a]$. But, by our calculations above and the properties of the a -norm,

$$\begin{aligned}
\left\| \sum_{n=k}^m \langle f, g_n \rangle_a g_n \right\|_{L^2(\mathbb{R})}^2 &= \int_0^a \left\| \sum_{n=k}^m \langle f, g_n \rangle_a g_n \right\|_a^2(t) dt \\
&= \int_0^a \left\langle \sum_{n=k}^m \langle f, g_n \rangle_a g_n, \sum_{n=k}^m \langle f, g_n \rangle_a g_n \right\rangle_a(t) dt \\
&= \int_0^a \sum_{n=k}^m |\langle f, g_n \rangle_a|^2(t) dt.
\end{aligned}$$

Now, $\sum_{n \in \mathbb{N}} |\langle f, g_n \rangle_a|^2$ converges in $L^1[0, a]$ implies that the right hand side of our equality goes to zero as $k \rightarrow \infty$.

The “moreover” part of the theorem follows immediately from Theorem 7.3 below. \square

5. A-FACTORABLE OPERATORS

Now we consider operators on $L^2(\mathbb{R})$ which behave naturally with respect to the a -inner product. We will call these operators a -factorable operators.

Definition 5.1. Fix $E \subset \mathbb{R}$ and $1 \leq p \leq \infty$. We say that a linear operator $L : L^2(\mathbb{R}) \rightarrow L^p(E)$ is an **a -factorable operator** if for any factorization $f = \phi g$ where $f, g \in L^2(\mathbb{R})$ and ϕ is an a -periodic function we have

$$L(f) = L(\phi g) = \phi L(g).$$

First we show it is enough to consider factorizations over $L^\infty([0, a])$

Proposition 5.2. Let T be a bounded operator from $L^2(\mathbb{R})$ to $L^2(E)$. Then T is a -factorable if and only if $T(\phi f) = \phi T(f)$ for all $f \in L^2(\mathbb{R})$ and all a -periodic $\phi \in L^\infty(\mathbb{R})$.

Proof. Assume ϕ is a -periodic, $f, g \in L^2(\mathbb{R})$ and $f = \phi g$. For all $n \in \mathbb{N}$ let

$$F_n = \{t \in [0, a] : |\phi(t)| > n\}.$$

Let $E_n = [0, 1] - F_n$ and

$$\tilde{E}_n = \cup_{m \in \mathbb{Z}} (E_n + m) \quad \text{and} \quad \tilde{F}_n = \cup_{m \in \mathbb{Z}} (F_n + m).$$

Now,

$$\begin{aligned} \|\chi_{\tilde{E}_n} \phi g - \phi g\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |\chi_{\tilde{E}_n} \phi(t) g(t)|^2 dt \\ &= \int_0^a |\chi_{F_n} \phi(t)|^2 < g, g >_a(t) dt. \end{aligned}$$

Since $\phi g \in L^2(\mathbb{R})$ and $\lim_{n \rightarrow \infty} \lambda(F_n) = 0$, it follows that $h_n =: \chi_{\tilde{E}_n} \phi g$ converges to ϕg in $L^2(\mathbb{R})$. Since T is a bounded linear operator, it follows that $T(h_n)$ converges to $T(\phi g)$. But, $T(h_n) = \chi_{\tilde{E}_n} \phi T(g)$ by our assumption. Now,

$$\|T(h_n)\| \leq \|T\| \|h_n\| \leq \|T\| \|\phi g\| = \|T\| \|f\|.$$

Finally, since $|T(h_n)| \uparrow |\phi T(g)|$ it follows from the Lebesgue Dominated Convergence Theorem that $\phi T(g) \in L^2(\mathbb{R})$ and $T(h_n) \rightarrow \phi T(g)$. This completes the proof of the Proposition. \square

We have immediately,

Corollary 5.3. *An operator $T : L^2(\mathbb{R}) \rightarrow L^p(E)$ is a -factorable if and only if $T(E_{m/a}g) = E_{m/a}T(g)$, for all $m \in \mathbb{Z}$. That is, T is a -factorable if and only if it commutes with $E_{m/a}$.*

The a -inner product naturally defines several types of a -linear maps. We present two of them here.

Proposition 5.4. *Fix $g \in L^2(\mathbb{R})$ and define a linear operator $L : L^2(\mathbb{R}) \rightarrow L^1[0, 1]$ by*

$$L(f) = < f, g >_a.$$

Then L is a bounded, linear a -factorable operator with

$$\|L\| = \|g\|_{L^2(\mathbb{R})}.$$

Proof. We have that L is a -factorable by Proposition 3.6. Now, for any $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned}
\|Lf\| &= \| \langle f, g \rangle_a \|_{L^1[0,a]} \\
&= \int_0^a \left| \sum_{n \in \mathbb{Z}} f(t-na) \overline{g(t-na)} \right| dt \\
&\leq \int_0^a \sqrt{\sum_{n \in \mathbb{Z}} |f(t-na)|^2} \sqrt{\sum_{n \in \mathbb{Z}} |g(t-na)|^2} dt \\
&\leq \left(\int_0^a \sum_{n \in \mathbb{Z}} |f(t-na)|^2 \right)^{1/2} \left(\int_0^a \sum_{n \in \mathbb{Z}} |g(t-na)|^2 \right)^{1/2} \\
&= \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Letting $g = f$ we see that $\|L(g)\| = \|g\|$ which, combined with the above, shows that $\|L\| = \|g\|$. \square

Now we define another natural class of a-factorable operators.

Proposition 5.5. *If $g \in L_a^\infty(\mathbb{R})$, the operator*

$$L(f) = \langle f, g \rangle_a,$$

is a bounded linear operator mapping $L^2(\mathbb{R})$ onto $L^2[0, a]$ and

$$\|L\|^2 = \text{ess sup}_{[0,a]} \langle g, g \rangle_a.$$

Proof. This follows directly from the first part of the proof of Proposition 4.12 and again, letting $g = f$ above gives the norm of the operator. \square

Now, let L be any a-factorable linear operator from $L^2(\mathbb{R})$ to $L^p(A)$, and let $E = \ker L$. If $f \in E$, and $\phi \in L_a^\infty(\mathbb{R})$, then $L(\phi f) = \phi L(f) = 0$. So $\phi f \in E$. We summarize this below.

Proposition 5.6. *If L is any a-factorable linear operator with kernel E , then E is an a-periodic closed set and so $E^\perp = E^{\perp_a}$.*

On more property of a-factorable operators into $L^2[0, a]$ is that the operator is bounded pointwise by its operator norm with respect to the a-norm.

Proposition 5.7. *Let L be an bounded a-factorable linear operator from $L^2(\mathbb{R})$ to $L^2[0, a]$. Then for all $f \in L^2(\mathbb{R})$ we have*

$$|L(f)(t)| \leq \|L\| \|f\|_a(t), \quad \text{for all } t \in [0, a].$$

Proof. If not, there is an $f \in L^2(\mathbb{R})$ and a set $B \subset [0, a]$ of positive measure so that

$$|L(f)(t)| > \|L\| \|f\|_a(t), \quad \text{for all } t \in B.$$

Case 1 If $\|f\|_a(t) = 0$ for a.e. $t \in B$. Let $\Phi = \sum_n T_{na} \mathbf{1}_B$ so $\Phi f = 0$ yet $L(\Phi f) \neq 0$ and we have our contradiction.

Case 2 If $\|f\|_a(t) \neq 0$ for a.e. $t \in B$ and let $A \subset B$ so that $\|f\|_a(t) \neq 0$ for $t \in A$. We define $\phi = \sum_n T_{na} \mathbf{1}_A$. Now $\phi f \in L^2(\mathbb{R})$ and

$$\left\| \frac{\phi f}{\|\phi f\|_a} \right\|_{L^2(\mathbb{R})}^2 \leq \lambda(A).$$

But,

$$\left\| L \left(\frac{\phi f}{\|\phi f\|_a} \right) \right\|_{L^2[0,a]}^2 \geq \int_A \left| L \left(\frac{\phi f}{\|\phi f\|_a} \right) (t) \right|^2 dt > \lambda(A) \|L\|^2,$$

which is a contradiction. \square

Now we present a short proof of the Riesz representation theorem for a -factorable operators from $L^2(\mathbb{R})$ to $L^1[0, a]$.

Theorem 5.8 (Riesz Representation Theorem). *Let L be a bounded a -factorable linear operator from $L^2(\mathbb{R})$ to $L^1[0, a]$. There exists a function $g \in L^2(\mathbb{R})$ such that $L(f) = \langle f, g \rangle_a$, for all $f \in L^2(\mathbb{R})$ and $\|L\| = \|g\|_{L^2(\mathbb{R})}$.*

Proof. Let $f \in L^2(\mathbb{R})$ and consider the a -orthonormal basis $g_n(x) = T_{na} \chi_{[0,a)}(x)$. Hence we have the decomposition

$$f = \sum_n \langle f, g_n \rangle_a g_n$$

We will show the function below is the one we are looking for.

$$g = \sum_{k \in \mathbb{Z}} \widetilde{L(g_k)} g_k,$$

where $\widetilde{L(g_k)}$ denotes the periodic extension of $L(g_k)$ to \mathbb{R} . First we must show this function is in $L^2(\mathbb{R})$. For positive integers n we define:

$$h_n = \sum_{|k| \leq n} \widetilde{L(g_k)} g_k.$$

For any $\phi \in L^2[0, a]$ we have

$$\langle \phi, L(g_k) \rangle = \int_0^a \phi(t) L(g_k)(t) dt = \int_0^a L(\tilde{\phi} g_k)(t) dt \leq \|\phi\|_{L^2[0,a]} \|L\|$$

Since ϕ was arbitrary, $L(g_k) \in L^2[0, a]$ and thus $\widetilde{L(g_k)} g_k \in L^2(\mathbb{R})$. It follows that $h_n \in L^2(\mathbb{R})$. Note that $\|h_n\|_{L^2(\mathbb{R})}^2 = \sum_{|k| \leq n} \|L(g_k)\|_{L^2[0,a]}^2$. Now we compute

$$\begin{aligned}
\left\| L \left(\frac{h_n}{\|h_n\|_{L^2(\mathbb{R})}} \right) \right\|_{L^1[0,a]} &= \frac{1}{\|h_n\|_{L^2(\mathbb{R})}} \int_0^a \sum_{|k| \leq n} |L(g_k)|^2(t) dt \\
&= \|h_n\|_{L^2(\mathbb{R})} \leq \|L\|.
\end{aligned}$$

Since n was arbitrary it follows that $g \in L^2(\mathbb{R})$.

A direct calculation shows that this is the correct g . i.e. For all $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned}
\langle f, g \rangle_a &= \left\langle \sum_n \langle f, g_n \rangle_a g_n, \sum_k \overline{L(g_k)} g_k \right\rangle_a \\
&= \sum_n \langle f, g_n \rangle_a L(g_n) = L(f)
\end{aligned}$$

□

Without much difficulty one may extend this characterization to a-factorable operators on other $L^p(\mathbb{R})$ spaces as well as into other $L^p[0, a]$ spaces. We state one of these because it will be of use in applications to Weyl-Heisenberg frames.

Proposition 5.9. *Let L be a bounded a-factorable linear operator from $L^2(\mathbb{R})$ to $L^2[0, a]$. There exists a function $g \in L^2(\mathbb{R})$ such that $L(f) = \langle f, g \rangle_a$, for all $f \in L^2(\mathbb{R})$.*

Proof. We note that $L^2[0, a] \subset L^1[0, a]$ and apply the same proof as above only now it is clear that $h_n \in L^2(\mathbb{R})$. □

We end this section by verifying that for a-factorable operators T , the operator T^* behaves as it should relative to the a-inner product.

Proposition 5.10. *If T is an a-factorable operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, then for all $f, g \in L^2(\mathbb{R})$ we have*

$$\langle T^*(f), g \rangle_a = \langle f, T(g) \rangle_a.$$

Proof. Since the a-inner product is a-periodic, we only need to show the above equality with these functions restricted to $L^2[0, a]$. For all $m \in \mathbb{Z}$ we have,

$$\begin{aligned}
\langle T^*(f), E_{m/a} g \rangle &= \int_{\mathbb{R}} T^*(f)(t) \overline{g(t)} e^{-2\pi i(m/a)t} dt \\
&= \int_0^a \langle T^*(f), g \rangle_a(t) e^{-2\pi i(m/a)t} dt.
\end{aligned}$$

Also,

$$\begin{aligned}
\langle f, T(E_{m/a}g) \rangle &= \langle f, E_{m/a}T(g) \rangle \\
&= \int_{\mathbb{R}} f(t) \overline{T(g)(t)} e^{-2\pi i(m/a)t} dt \\
&= \int_0^a \langle f, T(g) \rangle_a e^{-2\pi i(m/a)t} dt.
\end{aligned}$$

Since $\langle f, T(E_{m/a}g) \rangle = \langle T^*(f), E_{m/a}g \rangle$, for all $m \in \mathbb{Z}$, it follows from the above that,

$$\int_0^a \langle T^*(f), g \rangle_a e^{-2\pi i(m/a)t} dt = \int_0^a \langle f, T(g) \rangle_a e^{-2\pi i(m/a)t} dt,$$

for all $m \in \mathbb{Z}$. But, this means that

$$\langle \langle T^*(f), g \rangle_a, e^{-2\pi i(m/a)t} \rangle = \langle \langle f, T(g) \rangle_a, e^{-2\pi i(m/a)t} \rangle,$$

for all $m \in \mathbb{Z}$, where the outer inner product is taken in $L^2[0, a]$. Since $(\frac{1}{\sqrt{a}}e^{-2\pi i(m/a)t})_{m \in \mathbb{Z}}$ is an orthonormal basis for $L^2[0, a]$, we get the desired equality. \square

6. WEYL-HEISENBERG FRAMES AND THE A-INNER PRODUCT

Now we apply our a-inner product theory to Weyl-Heisenberg frames. For any WH-frame (g, a, b) , it is well known that the frame operator S commutes with E_{mb}, T_{na} . Thus, Corollary 5.3 yields:

Corollary 6.1. *If (g, a, b) is a WH-frame, then the frame operator S is a $1/b$ -factorable operator.*

We next show that the WH-Frame Identity for (g, a, b) has an interesting representation in both the a and the $\frac{1}{b}$ inner products. The known WH frame identity requires that the function f be bounded and of compact support. While this remains a condition for the WH-Frame Identity derived from the a -inner product we are able to extend this result to all $f \in L^2(\mathbb{R})$ when we use the $\frac{1}{b}$ -inner product. For this reason we present the theorems separately.

The proof of both these theorems have their roots in the Heil and Walnut proof of the WH-Frame Identity (see [19], Theorem 4.1.5). We refer the reader to Proposition 3.1 and Corollary 3.2 for questions concerning convergence of the series and integrals below.

Theorem 6.2. *Let $g \in L_a^\infty(\mathbb{R})$, and $a, b \in \mathbb{R}^+$. For all $f \in L^2(\mathbb{R})$ which are bounded and compactly supported we have*

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = b^{-1} \sum_k \int_0^a \langle T_{k/b} f, f \rangle_a \langle g, T_{k/b} g \rangle_a dt.$$

Proof. We start with the WH-frame Identity realizing that $\langle g, T_{k/b} g \rangle_a$ is a -periodic.

$$\begin{aligned} & \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = \\ &= b^{-1} \sum_k \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) \sum_n g(t - na) \overline{g(t - na - k/b)} dt \\ &= b^{-1} \sum_k \sum_j \int_0^a \overline{f(t - ja)} f(t - k/b - ja) \langle g, T_{k/b} g \rangle_a dt \\ &= b^{-1} \sum_k \int_0^a \langle T_{k/b} f, f \rangle_a \langle g, T_{k/b} g \rangle_a dt \end{aligned}$$

□

For the rest of this section we concentrate on the $\frac{1}{b}$ inner product and its relationship to WH-frames. In a forthcoming paper on the WH-Frame identity we more closely examine the role of the a -inner product. We also show in this paper that one may relax the condition on g . That is, the original WH-frame identity holds for all $g \in L^2(\mathbb{R})$ when f is bounded and compactly supported.

Theorem 6.3. *Let $g \in L_a^\infty(\mathbb{R})$, and $a, b \in \mathbb{R}^+$. For all $f \in L^2(\mathbb{R})$ we have*

$$\sum_{m \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = \| \langle f, T_{na} g \rangle_{1/b} \|_{L^2[0,1/b]}^2,$$

and so

$$\sum_{n,m \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = \sum_{n \in \mathbb{Z}} \| \langle f, T_{na} g \rangle_{1/b} \|_{L^2[0,1/b]}^2.$$

Proof. We just compute,

$$\begin{aligned}
\sum_{m \in \mathbb{Z}} | \langle f, E_{mb} T_{na} g \rangle |^2 &= \sum_{m \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(t) \overline{g(t - na)} e^{-2\pi i m b t} dt \right|^2 \\
&= b^{-1} \sum_{m \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \int_0^{1/b} f(t - k/b) \overline{g(t - na - k/b)} e^{-2\pi i m b t} dt \right|^2 \\
&= b^{-1} \sum_{m \in \mathbb{Z}} \left| \int_0^{1/b} \langle f, T_{na} g \rangle_{\frac{1}{b}}(t) e^{-2\pi i m b t} dt \right|^2 \\
&= b^{-1} \int_0^{1/b} | \langle f, T_{na} g \rangle_{\frac{1}{b}}(t) |^2 dt \\
(6.1) \quad &= \| \langle f, T_{na} g \rangle_{1/b} \|_{L^2[0, 1/b]}^2.
\end{aligned}$$

□

Comparing the equality from Theorem 6.3 above to the frame inequalities we have,

Corollary 6.4. *Let $g \in \mathbf{PF}$, $a, b \in \mathbb{R}^+$ and define $L : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by*

$$L(f) = \sum_{n \in \mathbb{Z}} b^{-1} \langle f, T_{na} g \rangle_{\frac{1}{b}} \chi_{[n/b, (n+1)/b]}.$$

We have

$$\|L(f)\|^2 = \sum_{m, n \in \mathbb{Z}} | \langle f, E_{mb} T_{na} g \rangle |^2.$$

Hence, L is a bounded linear operator which is an isomorphism if and only if (g, a, b) is a WH-frame. Moreover, if (g, a, b) has frame bounds A, B , then $\sqrt{A}\|f\| \leq \|L(f)\| \leq \sqrt{B}\|f\|$, for all $f \in L^2(\mathbb{R})$. Hence, (g, a, b) is a normalized tight frame if and only if L is an isometry.

Now we want to directly relate our a -inner product to WH-frames.

Proposition 6.5. *If $g, h \in L_{1/b}^\infty(\mathbb{R})$, then for all $f \in L^2(\mathbb{R})$ we have*

$$\sum_{m \in \mathbb{Z}} \langle f, E_{mb} g \rangle E_{mb} h = \frac{1}{b} \langle f, g \rangle_{1/b} h,$$

where the series converges unconditionally in $L^2(\mathbb{R})$. Hence, $\langle f, g \rangle_{1/b} g \in \text{span} (E_{mb} g)_{m \in \mathbb{Z}}$.

Proof. By Proposition 5.5 we know that $\langle f, g \rangle_{1/b} \in L^2[0, 1/b]$. Next, for any $m \in \mathbb{Z}$ we have

$$\langle f, E_{mb} g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t)} e^{-2\pi i m b t} dt = \int_0^{1/b} \langle f, g \rangle_{1/b}(t) e^{-2\pi i m b t} dt.$$

Therefore, if we restrict ourselves to $L^2[0, 1/b]$ we have

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \langle f, E_{mb}g \rangle E_{mb} &= \sum_{m \in \mathbb{Z}} \left(\int_0^{1/b} \langle f, g \rangle_{1/b} e^{-2\pi i m b t} \right) e^{2\pi i m b t} dt \\ &= \frac{1}{b} \sum_{m \in \mathbb{Z}} \left\langle \langle f, g \rangle_{1/b}, \sqrt{b} E_{mb} \right\rangle \sqrt{b} E_{mb} = \frac{1}{b} \langle f, g \rangle_{1/b}. \end{aligned}$$

Once we know that we have this convergence in $L^2[0, 1/b]$, then redoing the above on \mathbb{R} with h inserted proves the result and convergence in $L^2(\mathbb{R})$. \square

There are several interesting consequences of this proposition. First we recapture the following result due to de Boor, Devore, and Ron [2]

Corollary 6.6. *For $g \in L^2(\mathbb{R})$ and $b \in \mathbb{R}$, the orthogonal projection P of $L^2(\mathbb{R})$ onto $\text{span}(E_{mb}g)_{m \in \mathbb{Z}}$ is*

$$Pf = \frac{1}{\|g\|_{1/b}^2} \langle f, g \rangle_{1/b} g,$$

where if $\|g\|_{1/b}(t) = 0$ then $g(t) = 0$ so we interpret $\frac{g(t)}{\|g\|_{1/b}^2(t)} = 0$.

Proof. By Proposition 4.9, we have that $(\sqrt{b} E_{mb} \frac{g}{\|g\|_{1/b}})_{m \in \mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$. Hence, for all $f \in L^2(\mathbb{R})$ we have by Proposition 6.5

$$\begin{aligned} Pf &= \sum_{m \in \mathbb{Z}} \langle f, \sqrt{b} E_{mb} \frac{g}{\|g\|_{1/b}} \rangle \sqrt{b} E_{mb} \frac{g}{\|g\|_{1/b}} = \\ &= b \sum_{m \in \mathbb{Z}} \langle f, E_{mb} \frac{g}{\|g\|_{1/b}} \rangle E_{mb} \frac{g}{\|g\|_{1/b}} \\ &= \langle f, \frac{g}{\|g\|_{1/b}} \rangle_{1/b} \frac{g}{\|g\|_{1/b}} = \frac{1}{\|g\|_{1/b}^2} \langle f, g \rangle_{1/b} g. \end{aligned}$$

\square

Combining Theorem 4.13 and Corollary 6.6 we have:

Proposition 6.7. *If $(g_n)_{n \in \mathbb{Z}}$ is a $1/b$ -orthonormal sequence in $L^2(\mathbb{R})$, then*

$$P(f) = \sum_{n \in \mathbb{Z}} \langle f, g_n \rangle_{1/b} g_n,$$

is the orthogonal projection of $L^2(\mathbb{R})$ onto $\text{span}(E_{mb}g_n)_{n, m \in \mathbb{Z}}$

In a paper devoted to the study shift-invariant frames and shift-invariant Riesz bases [25], Ron and Shen develop a powerful technique called **fiberization** to decompose the preframe operator and its adjoint into a simple collection of

operators called fibers. They then go on to apply this technique to Weyl-Heisenberg frames by considering the shift invariant space generated by the countable set $\{E_{mb}g\}_{m \in \mathbb{Z}}$ [26]. This allows them to produce their amazing result regarding the duality principle and Weyl-Heisenberg frames. By using the $1/b$ -inner product we are able to avoid many of the complicated lattice and dilation arguments needed for fiberization. In doing so we produce the type of fiberization of the frame operator for a general system that they have for the self adjoint system. Finally we note that all of our results have been done on the space side therefore eliminating any need for taking inverse Fourier transforms to represent the frame operator. What results is a simple fiber representation of the WH-frame operator which we refer to as a **compression of the frame operator**.

Theorem 6.8. *If (g, a, b) is a WH-frame with frame operator S , then S has the form*

$$S(f) = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{1/b} T_{na}g = \frac{1}{b} \sum_{n \in \mathbb{Z}} P_n f \cdot T_{na} \|g\|_{1/b}^2,$$

where P_n is the orthogonal projection of $L^2(\mathbb{R})$ onto $\text{span} (E_{mb}T_{na}g)_{m \in \mathbb{Z}}$ and the series converges unconditionally in $L^2(\mathbb{R})$.

Proof. If (g, a, b) is a WH-frame then by Proposition 2.8 we have that $\langle g, g \rangle_{1/b} \leq B$ a.e. Now, by definition of the frame operator S we have

$$\begin{aligned} S(f) &= \sum_{m, n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}g \\ &= \sum_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}g \right) \\ &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na}g \rangle_{1/b} T_{na}g. \end{aligned}$$

An application of Corollary 6.6 and Theorem 3.4 (10) completes the proof. \square

This simple representation of the frame operator converges “super-fast”. That is, we do not have to compute any of the modulation parameters to get $S(f)$. While this has immediately become a useful tool for deriving new properties regarding the frame operator, because this compression requires us to pointwise evaluate infinite sums of functions it has obvious shortcomings in applications.

Theorem 2.11 is a classification of the normalized tight WH-frames. We can now restate this in terms of the a -inner products.

Proposition 6.9. *Let (g, a, b) be a WH-frame. the following are equivalent:*

- (1) $(E_{mb}T_{na}g)_{n,m \in \mathbb{Z}}$ is a normalized tight Weyl-Heisenberg frame.
- (2) $(\frac{1}{\sqrt{b}}T_{n/b}g)_{n \in \mathbb{Z}}$ is an orthonormal sequence in the a -inner product.
- (3) We have that $g \perp_a T_{k/b}g$, for all $k \neq 0$ and $\langle g, g \rangle_a = b$ a.e.

Putting Corollary 6.7 and Proposition 6.9 together we have

Corollary 6.10. *If (g, a, b) is a normalized tight Weyl-Heisenberg frame, then*

$$P(f) = \frac{1}{b^2} \sum_{n \in \mathbb{Z}} \langle f, T_{n/b}g \rangle_a T_{n/b}g$$

is the orthogonal projection of $L^2(\mathbb{R})$ onto $\text{span} (E_{m/a}T_{k/b}g)_{n,m \in \mathbb{Z}}$.

7. TWO THEOREMS ON WH-FRAMES

In this section we will use the theory developed above to: (1) Classify the $g \in L^2(\mathbb{R})$ for which (g, a, b) is complete when $ab = 1$; and (2) Give an equivalent formulation of the necessary condition for (g, a, b) to form a WH-frame given in Theorem 2.7.

First, we need some notation. If $g \in L^2(\mathbb{R})$ and $a > 0$ let

$$X_{g,a} = \overline{\text{span}} (E_{ma}g)_{m \in \mathbb{Z}}.$$

If $A \subset [0, a]$ and $\phi \in L^2(A)$, we write $\tilde{\phi}$ for the a -periodic extension of ϕ to all of \mathbb{R} . If $g \in L_a^\infty(\mathbb{R})$, let

$$g\tilde{L}^2[0, a] = \{\tilde{\phi}g : \phi \in L^2[0, a]\}.$$

Lemma 7.1. *Let $E \subset [0, a]$ and $g \in L^2(\mathbb{R})$ and $A, B > 0$. The following are equivalent:*

- (1) $A \leq \langle g, g \rangle_a \leq B$ a.e. on E .
- (2) $A\|\phi\|^2 \leq \|\tilde{\phi}g\|^2 \leq B\|\phi\|^2$, for all $\phi \in L^2(E)$.

Proof. If $E \subset [0, a]$, and $\phi \in L^2(E)$ then

$$\|\tilde{\phi}g\|^2 = \int_{\mathbb{R}} |\tilde{\phi}|^2 |g|^2 dt = \int_E |\phi|^2 \langle g, g \rangle_a dt.$$

Rephrasing this, we have

$$A \int_E |\phi|^2 dt \leq \int_E |\phi|^2 \langle g, g \rangle_a dt \leq B \int_E |\phi|^2 dt, \text{ for all } \phi \in L^2[0, a].$$

The result is immediate from here. \square

Proposition 7.2. *Let $g \in L_a^\infty(\mathbb{R})$, $A, B > 0$. The following are equivalent:*

- (1) $A \leq \langle g, g \rangle_a \leq B$ a.e. on the support of $\langle g, g \rangle_a$.
- (2) $A\|\phi\|^2 \leq \|\tilde{\phi}g\|^2 \leq \|\phi\|^2$.
- (3) $X_{g,1/a} = g\tilde{L}^2[0, a]$.

Proof. The equivalence of (1) and (2) is Lemma 7.1.

(1) \Rightarrow (2): Let $h \in X_{g,1/a}$. Choose $h_n \in L^2[0, a]$

$$h_n = \sum_{|k| \leq n} a_k E_{k/a},$$

so that $\lim_{n \rightarrow \infty} h_n g = h$ in $L^2(\mathbb{R})$. Now, for all $m, n \in \mathbb{Z}$ we have

$$\begin{aligned} \|h_n - h_m\|_{L^2[0,a]}^2 &= \int_0^a |h_n(t) - h_m(t)|^2 dt \\ &\leq \frac{1}{A} \int_0^a |h_n(t) - h_m(t)|^2 < g, g >_a dt \\ &= \frac{1}{A} \int_{\mathbb{R}} |h_n(t)g(t) - h_m(t)g(t)|^2 dt \rightarrow 0, \text{ as } n, m \rightarrow \infty. \end{aligned}$$

Therefore, (h_n) is Cauchy in $L^2[0, a]$ and hence convergent to some $f \in L^2[0, a]$. Now,

$$\begin{aligned} \|h_n g - f g\|_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |h_n(t)g(t) - f(t)g(t)| dt \\ &= \int_0^a |h_n(t) - f(t)| < g, g >_a dt \\ &\leq B \int_0^a |h(t) - f(t)|^2 dt \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

That is, $f g = h \in g\tilde{L}[0, a]$.

(2) \Rightarrow (1): Define,

$$g_n(t) = \begin{cases} g(t) : & |g(t)| \leq n \\ 0 : & \text{otherwise.} \end{cases}$$

Let

$$E = (\text{supp } < g, g >_a) \cap [0, a].$$

Define $T_n : L^2(E) \rightarrow L^2(\mathbb{R})$ by $T_n(\phi) = \tilde{\phi} g_n$. For all n , T_n is a bounded linear operator and

$$\|T_n(\phi)\| = \|\tilde{\phi} g_n\| \leq \|\tilde{\phi} g\|,$$

Therefore, the (T_n) are pointwise bounded. By the Uniform Boundedness Principle, the (T_n) are uniformly bounded. Also,

$$|T_n \phi| \uparrow |\tilde{\phi} g| \in L^2(\mathbb{R}),$$

since $g\tilde{L}^2[0, a] = X_{g,1/a} \subset L^2(\mathbb{R})$. Hence, the operator T defined by $T(\phi) = \tilde{\phi} g$ is a bounded linear operator from $L^2(E)$ to $g\tilde{L}[0, a]$, which is one-to-one. Since

$X_{g,1/a}$ is a Banach space, it follows that T is an isomorphism. Hence, there are constants $A, B > 0$ satisfying,

$$A\|\phi\|^2 \leq \|\tilde{\phi}g\|^2 \leq B\|\phi\|^2.$$

That is,

$$A \int_E |\phi(t)|^2 dt \leq \int_E |\phi(t)|^2(t) \langle g, g \rangle_a dt \leq B \int_E |\phi(t)|^2 dt.$$

Hence, $A \leq \langle g, g \rangle_a \leq B$. \square

Now we have an important consequence of these results which is a the modulation version of the shift-invariant result of Ron and Shen [25] (Theorem 2.2.14).

Theorem 7.3. *For $g \in L^2(\mathbb{R})$, the following are equivalent:*

(1) *There are numbers $A, B > 0$ so that $A \leq \langle g, g \rangle_a \leq B$ a.e.*

(2) *$(E_{m/a}g)_{m \in \mathbb{Z}}$ is a Riesz basic sequence.*

Hence, if $ab = 1$, then (1) and (2) are equivalent to

(3) *$(E_{mb}g)_{m \in \mathbb{Z}}$ is a Riesz basic sequence with Riesz basis constants \sqrt{A}, \sqrt{B} .*

Proof. (1) \Rightarrow (2): In the proof of Proposition 7.2 we saw that the map $T : L^2[0, a] \rightarrow L^2(\mathbb{R})$ given by $T(\phi) = \tilde{\phi}g$ is an isomorphism. Since $(\frac{1}{\sqrt{a}}E_{m/a})_{m \in \mathbb{Z}}$ is an orthonormal basis for $L^2[0, a]$, it follows that $(T(E_{m/a})) = (E_{m/a}g)$ is a Riesz basic sequence.

(2) \Rightarrow (1): By assumption there are constants $A, B > 0$ satisfying for all sequences of scalars $(a_m)_{m \in \mathbb{Z}}$,

$$A \sum_{m \in \mathbb{Z}} |a_m|^2 \leq \left\| \sum_{m \in \mathbb{Z}} |a_m| E_{mb}g \right\|_{L^2(\mathbb{R})}^2 \leq B \sum_{m \in \mathbb{Z}} |a_m|^2.$$

Since $\left\| \sum_m |a_m| E_{mb} \right\|^2 = \sum_m |a_m|^2$, it follows that for all $\phi \in L^2[0, a]$, $A\|\phi\|^2 \leq \|\tilde{\phi}g\|^2 \leq B\|\phi\|^2$. \square

Note that Theorem 7.3 is really half of Theorem 2.10. This seems to indicate that there is “another half” someplace which produces the whole result. It would be interesting to find this. Note also that if $g(t) = e^{-t^2}$, then the Fourier transform of g is $\hat{g}(t) = \sqrt{2\pi} e^{-t^2/2}$. A direct calculation shows that there are constants $A, B > 0$ such that

$$A \leq \langle g, g \rangle_1 \leq B \quad \text{and} \quad A \leq \langle \hat{g}, \hat{g} \rangle_1 \leq B.$$

It follows that $(E_m \hat{g}) = (T_m^* g)$ is a Riesz basic sequence. So it follows that both $(E_m g)_{m \in \mathbb{Z}}$ and $(T_m g)_{m \in \mathbb{Z}}$ are Riesz basic sequences, despite the fact that (g, a, b) is not a WH-frame.

Next we will completely identify the functions $g \in L^2(\mathbb{R})$ for which (g, a, b) is complete in $L^2(\mathbb{R})$. We need one more piece of notation. If $(E_n)_{n \in \mathbb{Z}}$ is any orthonormal basis for $L^2[0, a]$, we let R denote the **right hand shift operator** given by $R(E_n) = E_{n+1}$, for all $n \in \mathbb{Z}$.

Proposition 7.4. *Let $E_n = e^{2\pi i n t} \in L^2[0, 1]$, for all $n \in \mathbb{Z}$, and let $f = \sum_{n \in \mathbb{Z}} a_n E_n \in L^2[0, 1]$. The following are equivalent:*

- (1) $(R^n f)_{n \in \mathbb{Z}}$ is complete in $L^2[0, 1]$.
- (2) $f \neq 0$, a.e. in $L^2[0, 1]$.

Proof. First we compute,

$$R(f) = R\left(\sum_{n \in \mathbb{Z}} a_n E_n\right) = \sum_{n \in \mathbb{Z}} a_n E_{n+1} = E_1 \sum_{n \in \mathbb{Z}} a_n E_n = E_1 f.$$

Note that for $h \in L^2[0, 1]$, we have that $h \perp E_n f$ if and only if $h \bar{f} \perp E_n$. Hence, $h \perp E_n f$, for all $n \in \mathbb{Z}$ if and only if $h \bar{f} = 0$, a.e. It follows that $(E_n f)_{n \in \mathbb{Z}}$ is complete (and hence $(R^n f)$ is complete) if and only if we have: Whenever $h \in L^2[0, 1]$ and $h \bar{f} = 0$, a.e., then $h = 0$ a.e. This is clearly equivalent to $f \neq 0$, a.e. \square

Now we can give the required classification. If $f(x, y)$ is a function of two variables, we write f_x for the function $f_x(y) = f(x, y)$ and f_y for the function $f_y(x) = f(x, y)$.

Theorem 7.5. *Let $a = b = 1$ and $g \in L^2(\mathbb{R})$. The following are equivalent:*

- (1) $(E_m T_n g)_{m, n \in \mathbb{Z}}$ is complete in $L^2(\mathbb{R})$.
- (2) *There is a function $f(x, y) : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ satisfying:*
 - (a) *For a.e. $y \in [0, 1]$, we have that $f_y \neq 0$, a.e.*
 - (b) *For all $y \in [0, 1]$, we have*

$$g(y + n) = \hat{f}_y(n), \quad \text{for all } n \in \mathbb{Z},$$

where $\hat{f}_y(n) = \langle f_y, E_n \rangle$.

Proof. (2) \Rightarrow (1): Suppose that $h \in L^2(\mathbb{R})$ and $h \perp \text{span } (E_{mb} T_{na} g)_{n, m \in \mathbb{Z}}$. Then by Proposition 4.5,

$$\langle h, T_{na} g \rangle_{1/b} = \langle f, T_{na} \rangle_a = 0, \quad \text{a.e. for all } n.$$

That is,

$$\sum_{k \in \mathbb{Z}} f(y - ka) \overline{g(y - (k - n)a)} = 0, \quad \text{a.e. for all } n.$$

Letting $h_y = \sum_k h(y - ka) e^{2\pi i k x}$ and $g_y = \sum_k g(y - ka) e^{2\pi i k x}$, we have that $g_y(x) = f_y(x)$. Also by the above we have that

$$h_y \perp R^n(f_y), \quad \text{for all } n \in \mathbb{Z}.$$

Hence, by Proposition 7.4, we have that $h_y = 0$ a.e. That is, $h(y - ka) = 0$, for all $k \in \mathbb{Z}$. Hence, $h = 0$ a.e. and it follows that (g, a, b) is complete.

(1) \Rightarrow (2): Define the function

$$f(x, y) = \sum_{k \in \mathbb{Z}} g(y - ka) e^{2\pi i k x}.$$

Then the above argument for (2) \Rightarrow (1) shows that $f(x, y)$ has the desired properties. \square

Recall [22] that a class of infinitely differentiable functions on \mathbb{T} is called **quasi-analytic** if the only function in the class which vanishes with all its derivatives at some point $t_0 \in \mathbb{T}$ is the function which vanishes identically. A direct calculation shows that functions in a quasi-analytic class can have at most a finite number of zeroes on \mathbb{T} . On page 113 of Katznelson [22] is the following theorem.

Theorem 7.6. (*Denjoy-Carleman*) *Given some $d, K > 0$ in \mathbb{R} , let*

$$E = \{f = \sum_{n \in \mathbb{N}} a_n e^{2\pi i n t} : |a_n| \leq K d^n, \text{ for all } n\}.$$

Then E is a quasi-analytic class.

Combining the above we have

Theorem 7.7. *Let $g \in L^2(\mathbb{R})$ and assume there exist $K, d > 0$ so that*

$$|g(t + n)| \leq K d^n, \text{ for all } t \in [0, a].$$

Then $(E_m T_n)_{m, n \in \mathbb{Z}}$ is complete in $L^2(\mathbb{R})$. In particular, $g(t) = e^{-ct^2}$, works for all $c > 0$.

Finally, we recall [15] that if f is an H^p -function then $\log|f(e^{i\theta})|$ is integrable unless $f(z) \equiv 0$. In particular, if f vanishes on a set of positive measure then it vanishes identically. One consequence of this is that if $m \in \mathbb{Z}$ and $f = \sum_{k=m}^{\infty} a_k E_k \in L^p[0, 1]$ and $a_i \neq 0$ for at least one $m \leq i < \infty$, then $f \neq 0$ a.e. Combining this with the proof of Theorem 7.5 (the proof of (2) \Rightarrow (1)) we have

Theorem 7.8. *Let $g \in L^2(\mathbb{R})$ be supported on a ray $[\alpha, \infty)$ (In particular, if g has compact support). The following are equivalent:*

- (1) *$(E_m T_n g)_{m, n \in \mathbb{Z}}$ is complete in $L^2(\mathbb{R})$.*
- (2) *$\sup_{n \in \mathbb{Z}} |g(x - n)| \neq 0$ a.e.*

8. FRAMES IN THE A-INNER PRODUCT

We will now look at the notion of frames and Riesz bases in the a-inner product.

Definition 8.1. We say that a sequence $f_n \in L^2(\mathbb{R})$ is a **a-Riesz basic sequence** if there is an a -orthonormal basis $(g_n)_{n \in \mathbb{Z}}$ and an a -factorable operator T on $L^2(\mathbb{R})$ with $T(g_n) = f_n$ so that T is invertible on its range. If T is surjective, we call (f_n) a **a-Riesz basis** for $L^2(\mathbb{R})$.

Proposition 8.2. For $f_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$, the following are equivalent:

- (1) $(f_n)_{n \in \mathbb{Z}}$ is an a -Riesz basic sequence.
- (2) $(E_{m/a}f_n)_{n \in \mathbb{Z}}$ is a Riesz basic sequence.

Proof. (1) \Rightarrow (2): By assumption, there is an a -orthonormal basis (g_n) and an a -factorable operator T with

$$T(g_n) = f_n, \text{ for all } n \in \mathbb{Z}.$$

By the definition of an a -orthonormal basis we have that $(\frac{1}{\sqrt{a}}E_{m/a}g_n)_{m,n \in \mathbb{Z}}$ is an orthonormal basis $L^2(\mathbb{R})$. Since T is an isomorphism, it follows that

$$(T(\frac{1}{\sqrt{a}}E_{m/a}g_n))_{n,m \in \mathbb{Z}} = (\frac{1}{\sqrt{a}}E_{m/a}T(g_n))_{n,m \in \mathbb{Z}} = (\frac{1}{\sqrt{a}}E_{m/a}f_n)_{n,m \in \mathbb{Z}}$$

is a Riesz basic sequence.

(2) \Rightarrow (1): Let $g = \chi_{[0,a)}$ so that $(\frac{1}{\sqrt{a}}E_{m/a}T_n g)_{m,n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. Then

$$T(\frac{1}{\sqrt{a}}E_{m/a}T_n g) = E_{m/a}f_n$$

is an a -factorable linear operator which is an isomorphism because $(E_{m/a}f_n)$ is a Riesz basic sequence. Hence, (f_n) is an a -Riesz basic sequence. \square

Corollary 8.3. For $f_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$, the following are equivalent:

- (1) $(f_n)_{n \in \mathbb{Z}}$ is an a -Riesz basis.
- (2) $(E_{m/a}f_n)_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$.

Since the inner product on a Hilbert space is used to define a frame, we can get a corresponding concept for the a -inner product.

Definition 8.4. If $g_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$, we call $(g_n)_{n \in \mathbb{Z}}$ an **a-frame sequence** if there exist constants $A, B > 0$ so that for all $f \in \text{span} (E_{m/a}g_n)_{m,n \in \mathbb{Z}}$ we have

$$A\|f\|_a^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, g_n \rangle_a|^2 \leq B\|f\|_a^2.$$

If the inequality above holds for all $f \in L^2(\mathbb{R})$ then we call g_n an **a-frame**.

Now we have the corresponding result to Theorem 2.2.

Theorem 8.5. *Let $g_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$. The following are equivalent:*

- (1) $(g_n)_{n \in \mathbb{Z}}$ is an a -frame.
- (2) If $(e_n)_{n \in \mathbb{Z}}$ is an a -orthonormal basis for $L^2(\mathbb{R})$, and $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ with $T(e_n) = g_n$ is a -factorable, then T is a bounded, linear surjective operator on $L^2(\mathbb{R})$.

Proof. If $T(e_n) = g_n$, then

$$\langle T^*(f), e_n \rangle_a = \langle f, T(e_n) \rangle_a = \langle f, g_n \rangle_a.$$

Hence, by Theorem 4.13 we have that $T^*(f) = \sum_{n \in \mathbb{Z}} \langle f, g_n \rangle_a e_n$ and

$$\|T^*(f)\|_a^2 = \sum_{n \in \mathbb{Z}} |\langle f, g_n \rangle_a|^2.$$

Hence, (g_n) is an a -frame sequence if and only if

$$A\|f\|_a^2 \leq \|T^*(f)\|_a^2 \leq B\|f\|_a^2, \quad \text{for all } f \in L^2(\mathbb{R}).$$

But this is equivalent to T^* being an isomorphism, which itself is equivalent to T being a bounded, linear onto operator. \square

Finally, we can relate this back to our regular frame sequences.

Proposition 8.6. *Let $g_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$. The following are equivalent:*

- (1) $(g_n)_{n \in \mathbb{Z}}$ is an a -frame sequence.
- (2) $(E_{m/a}g_n)_{m,n \in \mathbb{Z}}$ is a frame sequence.

Proof. (1) \Rightarrow (2): If (g_n) is an a -frame sequence, then there is an a -orthonormal basis (e_n) for $L^2(\mathbb{R})$ and an a -factorable onto (closed range) operator $T(e_n) = g_n$. Now, $(E_{m/a}e_n)_{n,m \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$ and

$$T(E_{m/a}e_n) = E_{m/a}T(e_n) = E_{m/a}g_n.$$

Hence, $(E_{m/a}g_n)_{m,n \in \mathbb{Z}}$ is a frame sequence.

(2) \Rightarrow (1): Reverse the steps in part I above. \square

The following Corollary is immediate from Theorem 8.5 and Proposition 8.6.

Corollary 8.7. *Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. The following are equivalent:*

- (1) (g, a) is a $1/b$ -frame.
- (2) (g, a, b) is a Weyl-Heisenberg frame.

9. GRAM-SCHMIDT PROCESS

In this section we will look at the Gram-Schmidt process for the a -inner product. First we need a result which shows that this process produces functions which are in the proper spans.

Proposition 9.1. *Let $f, g, h \in L^2(\mathbb{R})$. We have:*

(1) $N_a(g) \in \text{span} (E_{m/a}g)_{m \in \mathbb{Z}}$.

(2) *If any two of f, g, h are in $L_a^\infty(\mathbb{R})$, then $\langle f, h \rangle_a g \in \text{span} (E_{m/a}g)_{m \in \mathbb{Z}}$.*

Proof.

(1): For each $n \in \mathbb{N}$ let

$$E_n = \{t \in [0, a] : |\langle g, g \rangle_a(t)|^2 \geq n \text{ or } \langle g, g \rangle_a(t) \leq \frac{1}{n}\}.$$

Also, let

$$\tilde{E}_n = \cup_{m \in \mathbb{Z}} (E_n + m).$$

Since $g \in L^2(\mathbb{R})$, we have

$$\|g\|^2 = \int_0^a \langle g, g \rangle_a(t) dt < \infty.$$

Hence, $\lim_{n \rightarrow \infty} \lambda(E_n) = 0$. Let $F_n = [0, a] - E_n$ and

$$\tilde{F}_n = \cup_{m \in \mathbb{Z}} (F_n + m).$$

Now,

$$\frac{1}{n} \leq \frac{1}{\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g \rangle_a} \leq n.$$

Hence,

$$\frac{1}{\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g \rangle_a} \in L_a^\infty(\mathbb{R}).$$

Hence,

$$\frac{\chi_{\tilde{F}_n} g}{\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g \rangle_a} + \chi_{\tilde{E}_n} g \in \text{span} (E_{m/a}g)_{m \in \mathbb{Z}}.$$

Also,

$$\begin{aligned} \left\| \frac{\chi_{\tilde{F}_n} g}{\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g \rangle_a} + \chi_{\tilde{E}_n} g - N_a(g) \right\|_{L^2(\mathbb{R})} &= \left\| \chi_{\tilde{E}_n} g - \frac{\chi_{\tilde{E}_n} g}{\langle g, g \rangle_a} \right\| \\ &\leq \|\chi_{\tilde{E}_n} g\| + \left\| \frac{\chi_{\tilde{E}_n} g}{\langle g, g \rangle_a} \right\| \\ &= \left(\int_{\mathbb{R}} |\chi_{\tilde{E}_n} g|^2 dt \right)^{1/2} + \|N_a(\chi_{\tilde{E}_n} g)\| \\ &\leq \left(\int_{E_n} \langle g, g \rangle_a(t) dt \right)^{1/2} + \lambda(E_n). \end{aligned}$$

But the right hand side of the above inequality goes to zero as $n \rightarrow \infty$.

(2): Assume first that $f, h \in L_a^\infty(\mathbb{R})$. Let $B = \|f\| + a$ and $C = \|h\|_a$. Now,

$$\begin{aligned}
| \langle f, h \rangle_a | &= \left| \sum_{n \in \mathbb{Z}} f(t - na) \overline{g(t - na)} \right| \\
&\leq \sqrt{\sum_{n \in \mathbb{Z}} |f(t - na)|^2} \sqrt{\sum_{n \in \mathbb{Z}} |g(t - na)|^2} \leq \sqrt{B} \sqrt{C}.
\end{aligned}$$

Therefore, $\langle f, h \rangle_a$ is a bounded a -periodic function on \mathbb{R} . this implies that $\langle f, h \rangle_a g \in L^2(\mathbb{R})$.

Now suppose that $g, h \in L_a^\infty(\mathbb{R})$. Let $B = \|g\|_a$ and $C = \|h\|_a$. Then

$$\begin{aligned}
\| \langle f, h \rangle_a g \|_{L^2(\mathbb{R})}^2 &= \left\| \left(\sum_{n \in \mathbb{Z}} f(t - na) \overline{h(t - na)} \right) g \right\|_{L^2(\mathbb{R})}^2 \\
&= \int_0^a \left| \sum_{n \in \mathbb{Z}} f(t - na) \overline{h(t - na)} \right|^2 \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt \\
&\leq B \int_0^a \sum_{n \in \mathbb{Z}} |f(t - na)|^2 \sum_{n \in \mathbb{Z}} |h(t - na)|^2 dt \\
&\leq BC \|f\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Recall that

$$\text{span} (E_{m/a} g)_{m \in \mathbb{Z}} = \{ \phi g : \phi \text{ is } a\text{-periodic and } \phi g \in L^2(\mathbb{R}) \}.$$

So by the above, we have that $\langle f, h \rangle_a g \in \text{span} (E_{m/a} g)_{m \in \mathbb{Z}}$. \square

Definition 9.2. Let $g_n \in L^2(\mathbb{R})$, for $1 \leq n \leq k$. We say that $(g_n)_{n=1}^k$ is **a-linearly independent** if for each $1 \leq n \leq k$, $g_n \notin \text{span} (E_{m/a} g_i)_{m \in \mathbb{Z}; 1 \leq i \neq n \leq k}$. An arbitrary family is **a-linearly independent** if every finite sub-family is a -linearly independent.

Now we carry out the Gram-Schmidt process.

Theorem 9.3. (*Gram-Schmidt orthonormalization procedure*) Let $(g_n)_{n \in \mathbb{N}}$ be an a -linearly independent sequence in $L^2(\mathbb{R})$ for $a > 0$. Then there exists an a -orthonormal sequence $(e_n)_{n \in \mathbb{N}}$ satisfying for all $n \in \mathbb{N}$:

$$\text{span} (E_{m/a} g_k)_{m \in \mathbb{Z}, 1 \leq k \leq n} = \text{span} (E_{m/a} e_k)_{m \in \mathbb{Z}, 1 \leq k \leq n}.$$

Proof We proceed by induction. First let $e_1 = N_a(g_1)$. If $(e_i)_{i=1}^n$ have been defined to satisfy the theorem, let

$$e_{n+1} = N_a(g_{n+1} - \sum_{i=1}^n \langle g_{n+1}, e_i \rangle_a e_i).$$

Let

$$h = g_{n+1} - \sum_{i=1}^n \langle g_{n+1}, e_i \rangle_a e_i.$$

Note that $h \neq 0$ by our a-linearly independent assumption and Proposition 9.1. Now, for $1 \leq k \leq n$ we have

$$\begin{aligned} \langle e_{n+1}, e_k \rangle_a &= \frac{1}{\langle h, h \rangle_a} \left(\langle g_{n+1}, e_k \rangle_a - \sum_{i=1}^n \langle g_{n+1}, e_i \rangle_a \langle e_i, e_k \rangle_a \right) \\ &= \frac{1}{\langle h, h \rangle_a} (\langle g_{n+1}, e_k \rangle_a - \langle g_{n+1}, e_k \rangle_a \langle e_k, e_k \rangle_a) = 0. \end{aligned}$$

The statement about the linear spans follows from Proposition 9.1. □

REFERENCES

- [1] J. Benedetto, C. Heil and D. Walnut, *Differentiation and the Balian-Low Theorem*, Jour. Fourier Anal. and Appls., 1 No. 4 (1995) 355-402.
- [2] C. de Boor, R. DeVore and A. Ron, *Approximation from shift invariant subspaces of $L_2(\mathbb{R}^d)$* , Trans. Amer. Math. Soc., (1994) 341:787-806.
- [3] C. de Boor, R. DeVore and A. Ron, *The Structure of shift invariant spaces and applications to approximation theory*, J. Functional Anal. No. 119 (1994), 37-78.
- [4] P.G. Casazza, *the art of frame theory*, preprint.
- [5] P.G. Casazza, *Modern tools for Weyl-Heisenberg frame theory*, preprint.
- [6] P.G. Casazza and O. Christensen, *Weyl-Heisenberg frames for subspaces of $L^2(\mathbb{R})$* , preprint.
- [7] P.G. Casazza, O. Christensen, and A.J.E.M. Janssen, *Classifying tight Weyl-Heisenberg frames*, preprint.
- [8] P.G. Casazza, O. Christensen and A.J.E.M. Janssen, *Weyl-Heisenberg frames, translation-invariant systems, and the Walnut representation*, preprint.
- [9] I. Daubechies, *Time-frequency localization operators: a geometric phase space approach*. IEEE Trans. Inform. Theory, 34 (1988) 605-612.
- [10] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*. IEEE Trans. Inform. Theory, 36 (5) (1990) 961-1005.
- [11] I. Daubechies, *“Ten Lectures on Wavelets”*, CBMS-NSF regional conference series in Applied Math., Philadelphia (1992).
- [12] I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*. J. Math. Phys. 27 (1986) 1271-1283.
- [13] I. Daubechies, H. Landau and Z. Landau, *Gabor time-frequency lattices and the Wexler-Rax identity*, J. Fourier Anal. and Appls. (1) No. 4 (1995) 437-478.
- [14] R.J. Duffin and A.C. Schaeffer, *A class of non-harmonic Fourier series*. Trans. AMS 72 (1952) 341-366.
- [15] Peter L. Duren, *Theory of H^p Spaces*, Academic Press, New York (1970).
- [16] H.G. Feichtinger and T. Strohmer, eds, *Gabor Analysis and Algorithms: Theory and Applications*, Birkhauser, Boston (1998).

- [17] D. Gabor, *Theory of communications*. Jour. Inst. Elec. Eng. (London) 93 (1946) 429-457.
- [18] D. Han and D. Larson, *Frames, Bases and Group Representations*, to appear, Memoirs AMS.
- [19] C. Heil and D. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review, 31 (4) (1989) 628-666.
- [20] A.J.E.M. Janssen, *Signal analytic proofs of two basic results on lattice expansions*, Appl. Comp. Har. Anal. 1 (4) (1994) 350-354.
- [21] A.J.E.M. Janssen, *Duality and biorthogonality for Weyl-Heisenberg frames*, Jour. Fourier Anal. and Appl. 1 (4) (1995) 403-436.
- [22] Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley and Sons, Inc. New York (1968).
- [23] J. Ramanathan and T. Steger, *Incompleteness of sparse coherent states*, Appl. comp. Harm. Anal. 2 (1995) 148-153.
- [24] M.A. Rieffel, *Von Neumann algebras associated with pairs of lattices in Lie groups*, Math. Anal. 257 (1981) 403-418.
- [25] A. Ron and Z. Shen, *Frames and stable basis for shift-invariant subspaces of $L^2(\mathbb{R}^d)$* , Canadian J. Math. 47(1995),1051-1094
- [26] A. Ron and Z. Shen, *Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$* , Duke Math. J. 89 (1997) 237-282.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA, MO 65211, AND DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208

E-mail address: `pete@math.missouri.edu;lammers@math.sc.edu`